# Dipartimento di scienze economiche, aziendali, matematiche e statistiche "Bruno de Finetti"

# On the optimal design of participating life insurance contracts

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ABSTRACT[1]

In this paper we study how policyholders and equityholders contribute to the formation of a life insurance company issuing participating contracts. The structure of these contracts is stylized and features a guaranteed rate of return and a terminal bonus, as in the pioneering model by Briys and de Varenne (1994, 1997). Policyholders aim at maximizing their preferences by choosing the leverage ratio and the guaranteed level, while being subject to regulatory constraints of fair valuation and solvency. We provide conditions under which non trivial contracts exist and analyze their properties.

KEYWORDS: Optimal contracts; Participating life insurance; Default risk; Mortality risk; Fairness

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# 1. Introduction

*Participating* life insurance policies (or policies *with profits*) have been very popular for a long time. Under these policies, the profits made by the insurance company through the investment of collected premiums are shared with the policyholders. This feature is often coupled with a minimum guarantee provision, that can be a *point-to-point* guarantee or a *cliquet*-style one, in which the annual profits are locked-in and not compensated with losses. Usually, the cost of the guarantees is not explicitly charged, but it is recovered through the participation mechanism. In fact, in 'good' years not all financial profits are credited to policyholders, but a fraction of them is retained by the insurer in order to fulfill the minimum guarantee promise in 'bad' years.

There are several ways in which the profit-sharing mechanism can be implemented. A very simple case prevailing in the Italian market, see Bacinello (2001, 2003a,b), is that of cliquet guarantees with annual returns recognized to policyholders depending only on the most recent performance of an investment portfolio. The contract valuation can the be reduced to that of a sequence of one-year forward-start options. In the UK market, annual returns recognized to policyholders depend are smoother as they depend on a certain number of past returns on the reference portfolio. Moreover, there is often a terminal bonus at maturity (see Wilkie (1987); Ballotta et al. (2006); Ballotta (2005)). More sophisticated smoothing adjustment mechanisms are in force in the Scandinavian countries (see Grosen and Jørgensen (2000), Hansen and Miltersen (2002), Miltersen and Persson (2003), Tanskanen and Lukkarinen (2003), Guillén et al. (2006)). Alternative features are present in the German market (see Kling et al. (2007a), Kling et al. (2007b)). Finally, several aspects concerning general bonus schemes, comparison between different participation mechanisms, study of the impact of surplus distribution strategies and management discretion are analysed in Norberg (1999), Norberg (2001), Aase (2002), Gatzert and Kling (2007), Gatzert (2008), Kleinow and Willder (2007), Kleinow (2009), Bohnert and Gatzert (2012), Bohnert et al. (2015).

As far as we are aware, Briys and de Varenne (1994, 1997) were the first to introduce the default risk in a stylized model of a participating life insurance contract. Their pioneering work has been extended and enhanced by a stream of papers, some of which have been cited above. We only mention the extensions due to Grosen and Jørgensen (2002), that introduce the possibility of early default in the model, and Bacinello et al. (2018), who introduce biometric risk at portfolio level.

According to the general philosophy of Briys and de Varenne (1994, 1997), a participating life insurance company can be seen as the result of an agreement between different stakeholders. The present paper is devoted to the study of their interaction leading to the formation of the insurance company. We start with an infinitely large population of policyholders who, together with an insurer (group of equity holders, seen as a whole), agree to form the company. Through the payment of a single premium each policyholder contributes to finance a fraction ('leverage ratio') of the assets of the company and, in return, is entitled to receive a benefit at maturity, if alive. We are then considering pure endowment contracts, often used in practice in order to build up a capital to be converted into a life annuity upon retirement. The benefit embeds a minimum guaranteed return and a surplus participation, but is subject to default risk in case the assets of the company

experience a poor performance. Each policyholder is exposed also to mortality risk (if she does not survive at the maturity date no payment is due), while equity holders are subject to systematic longevity risk (the number of policyholders alive at maturity is higher than what originally expected).

The variables negotiated between the two groups of stakeholders are the participation coefficient, the minimum guaranteed return and the leverage ratio. In particular, we assume that there is an immediate convergence on the participation coefficient, that is fixed according to the prevailing market conditions governing participating contracts, while policyholders aim at optimizing their preferences through the choice of guaranteed return and leverage ratio. Since policyholders are assumed to be perfectly homogeneous, we take the point of view of a representative, nonsatiated and risk averse, individual. On the other hand, equity holders are subject to regulatory constraints of fair pricing, maximum guarantees and solvency.

We observe that the optimization problem of the policyholder can be interpreted as an asset allocation problem between a risky asset (the insurance contract) and the riskless asset. In fact, the insurance contract is subject to default and mortality risk. However, policyholders who do not have bequest motives may perceive mortality risk as an opportunity and be willing to 'bet' on their survival for the purpose of achieving a benefit higher than that provided by a purely financial contract.

We show that the solution of the policyholder's optimization problem spans all non trivial contracts that are efficient, in a Pareto sense, for both stakeholders. We thoroughly inspect the problem and study conditions under which policyholders find the investment in the insurance contract attractive enough to allocate a positive share of their wealth. In particular, this occurs if the participation rate is sufficiently high to compensate the financial and mortality risk. Similarly, a too low ceiling for the minimum guaranteed rate will depress the demand for participating contracts.

There is a stream of papers in the actuarial literature which deal with the optimization of the contracts' design, rather than that of the capital structure for insurers, or other crucial aspects that affect the insurance business, among which we recall Mayers and Smith (1983), Berketi (1999), Chandra and Sherris (2005), Zhang (2008), Mao and Ostaszewski (2010), Laeven and Perotti (2010), Merz and Wüthrich (2014), Schmeiser and Wagner (2015), Braun et al. (2019), Chen and Hieber (2016). Schmeiser and Wagner (2015), Braun et al. (2019) and Chen and Hieber (2016) are notable examples that deal with a participating life insurance contract. Similarly to our work, the authors of the first two papers study an optimization problem, which aims at determining the interest rate guarantee level that should be set by the regulator in order to maximize the policyholder's utility. Both contributions use numerical simulations to claim that the optimal guarantee level is far below the maximum rate currently chosen by regulators. The third paper instead proposes a flexible regulatory supervision in order to lower the effect of a regulatory default constraint that might force the insurance company to offer Pareto inefficient contracts. The authors show that this problem can be alleviated if the regulator enforces an optimal investment strategy that decreases risk in case of distress, and that this strategy increases the policyholder's utility and leads to Pareto-improvements for both parties involved. The findings in these contributions are partially strengthened by the analytical framework introduced in our paper. More precisely, we establish general

results on the shape of optimal insurance contracts valid under nonrestrictive assumptions on policyholders' preferences, assets return distributions and stochastic mortality models.

The remainder of the paper is structured as follows. In Section 2 we describe our framework; Section 3, the core of the paper, contains our results concerning optimal contracts and Section 4 concludes the paper. Some properties and the proofs of our main results are collected in Appendices A, B, C.

## 2. Framework

A stylized formation of a participating life insurance company is as follows. Policyholders agree with equity holders on the share of total assets they will contribute to and the guaranteed benefit they will be entitled to, for a given participation policy and under some regulatory constraints.

We consider an infinitely large portfolio where all biometric risk has been pooled away and only systematic (longevity) risk matters. We act at individual (policyholder) level, and in order to do so we first focus on a finite portfolio and eventually let the portfolio size diverge.

Following Bacinello et al. (2018) and Briys and de Varenne (1994, 1997), we consider a cohort of $N_0$ homogeneous policyholders,[2] taking out at time 0 a guaranteed pure endowment contract with maturity $T$. The insurance company issues no other contracts or debt, raises no additional funds and pays no dividends. The capital allocated to each policyholder is $a_0 \equiv a_0(N_0) > 0$, and the percentage of such capital contributed by the policyholder through a single premium is $\alpha \in [0,1]$. The total assets of the company at time 0 therefore amount to $N_0 a_0$, of which the fraction $\alpha N_0 a_0$ is provided by policyholders and the remaining portion $(1-\alpha)N_0 a_0$ by equity holders.

We denote by $N$ the number of surviving policyholders at time $T$. As the contracts issued are pure endowments, only surviving policyholders are entitled to a benefit. The maturity payoff to the generic policyholder (liability for the insurer) is then

$$\ell^{(i)}(N_0) = 1_{E_i} \begin{cases} \frac{N_0}{N} a_0 \mathrm{e}^R & \text{if } \frac{N_0}{N} a_0 \mathrm{e}^R \leq \alpha a_0 G, \\ \alpha a_0 G & \text{if } \alpha a_0 G \leq \frac{N_0}{N} a_0 \mathrm{e}^R \leq a_0 G, \\ \alpha a_0 G + \delta \alpha a_0 \left( \frac{N_0}{N} \mathrm{e}^R - G \right) & \text{if } \frac{N_0}{N} a_0 \mathrm{e}^R \geq a_0 G, \end{cases}$$

$$= 1_{E_i} a_0 \left[ \alpha G + \delta \alpha \left( \frac{N_0}{N} \mathrm{e}^R - G \right)^+ - \left( \alpha G - \frac{N_0}{N} \mathrm{e}^R \right)^+ \right], \tag{1}$$

where $E_i$ is the event that the $i^{\text{th}}$ policyholder is alive at $T$, $R$ is the log-return of the company's assets over the period $[0,T]$, $\delta \in [0,1]$ is the participation rate and $G \geq 0$ is the guaranteed return. We assume that the guaranteed return $G$ is limited by an upper bound $\overline{G}$ imposed by the regulator; by convention, when $\overline{G} = +\infty$, no constraint is in force.[3] Note that the three components within the square brackets in (1) represent respectively the guaranteed benefit, the bonus (call) option stemming from the profit participation

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[2]In particular in terms of preferences and biometric risk profile.

[3]Information on maximum minimum guaranteed rates in different countries can be found, e.g., in Eling and Holder (2013)

and the default (put) option resulting from the shortage of assets with respect to the guaranteed benefit.

We let $\mathbb{P}$ be the physical measure and $\widetilde{\mathbb{P}}$ the pricing measure, equivalent to $\mathbb{P}$, chosen by the insurer. Expectations under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ will be denoted by $\mathbb{E}$ and, respectively, $\widetilde{\mathbb{E}}$. We assume that, under $\mathbb{P}$, the following assumptions hold: biometric and financial variables are independent; the log-return $R$ follows a distribution with support the entire real line; conditionally on a random variable $\Pi$ with support $(0,1)$, the survival events $(E_i)_{i=1,\dots,N_0}$ are, for all $N_0$, i.i.d. with common conditional probability

$$\mathbb{P}(E_i|\Pi) = \Pi.$$

The random variable $\Pi$ is the common, unknown, survival probability and therefore embodies the systematic longevity risk.

The change of measure from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ preserves the previous assumptions. In particular, $R$ has a distribution with support the entire real line and we assume the risk neutrality condition

$$\widetilde{\mathbb{E}}[e^R] = e^{rT}, \tag{2}$$

where $r$ is the constant risk-free rate. Moreover, we assume that, conditionally on a random variable $\widetilde{\Pi}$, the survival events $(E_i)_{i=1,\dots,N_0}$ are, for all $N_0$, i.i.d. with common conditional probability

$$\widetilde{\mathbb{P}}(E_i|\widetilde{\Pi}) = \widetilde{\Pi}.$$

The pricing measure $\widetilde{\mathbb{P}}$ incorporates a risk premium for financial and biometric risk. Indeed, the market requires that funds invested in a risky asset should earn on average a spread above the risk-free rate. On the other hand, insurers pricing assumptions will contain a loading for biometric (systematic or non systematic) risk, which in the present case should result in higher survival probabilities. We translate these facts into the following assumptions:

$$R \text{ under } \mathbb{P} \text{ dominates } R \text{ under } \widetilde{\mathbb{P}} \text{ in the usual stochastic order;} \tag{3}$$

$$\widetilde{\Pi} \text{ under } \widetilde{\mathbb{P}} \text{ dominates } \Pi \text{ under } \mathbb{P} \text{ in the usual stochastic order.} \tag{4}$$

Under Assumption (3) it holds in particular that

$$e^{rT} = \widetilde{\mathbb{E}}\left[e^R\right] \leq \mathbb{E}\left[e^R\right].$$

From Assumption (4) it follows that $\widetilde{\mathbb{E}}[\widetilde{\Pi}] \geq \mathbb{E}[\Pi]$ and in turn that, for all $i$,

$$\widetilde{\mathbb{P}}(E_i) \geq \mathbb{P}(E_i).$$

**Remark 2.1.** *A notable special case is that of normal return, when the distribution of $R$ under $\mathbb{P}$ is $N(\mu - \sigma^2/2, \sigma^2)$, with $\mu > r$, while under the risk neutral measure $\widetilde{\mathbb{P}}$ (unique when restricted to events relative to financial risk), is $N(r - \sigma^2/2, \sigma^2)$.*

Finally, we restrict the attention to contracts that are fairly priced and satisfy a solvency constraint. The contract is fair if its market price, i.e. the expected present value

under the pricing measure of the insurer's future liability, matches the initial premium: for all $i$, the following relationship holds.

$$\mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[\ell^{(i)}(N_0)\right] = \alpha a_0. \tag{5}$$

Note that fairness at individual level guarantees that the global insurer's future liabilities are fairly priced.

The solvency condition requires that, under the physical measure, the probability of default, i.e. the total assets of the company are insufficient to repay the global guaranteed benefit, cannot exceed a given confidence level $0 < \varepsilon < 1$ imposed by the regulator:

$$\mathbb{P}(a_0 N_0 \mathrm{e}^R \leq \alpha a_0 G N) \leq \varepsilon. \tag{6}$$

We now consider an infinitely large portfolio by taking the limit, as the portfolio size $N_0$ diverges, in the individual payoff (1), the fairness condition (5) and the solvency constraint (6).

For the individual payoff, we obtain

$$\ell^{(i)} = \lim_{N_0 \to +\infty} \ell^{(i)}(N_0)$$

$$= 1_{E_i} w_0 \left[\alpha G + \delta\alpha\left(\frac{\mathrm{e}^R}{\Pi} - G\right)^+ - \left(\alpha G - \frac{\mathrm{e}^R}{\Pi}\right)^+\right] \qquad \text{a.s., under } \mathbb{P} \tag{7}$$

$$= 1_{E_i} w_0 \left[\alpha G + \delta\alpha\left(\frac{\mathrm{e}^R}{\widetilde{\Pi}} - G\right)^+ - \left(\alpha G - \frac{\mathrm{e}^R}{\widetilde{\Pi}}\right)^+\right] \qquad \text{a.s., under } \widetilde{\mathbb{P}}, \tag{8}$$

where

$$w_0 = \lim_{N_0 \to +\infty} a_0(N_0) > 0$$

and we have exploited the fact that

$$\frac{N}{N_0} \to \Pi\,(\widetilde{\Pi}) \text{ a.s., under } \mathbb{P}\,(\text{under } \widetilde{\mathbb{P}}),$$

see Schervish (1995).[4] Hence, we have implicitly assumed the natural requirement that the capital allocated for each policyholder converges to a positive, finite limit $w_0$ as the portfolio size diverges. Hence, the limit $w_0$ can be interpreted as the individual capital allocated in a large portfolio.

For the fairness condition (5), taking the limit as $N_0 \to +\infty$ gives

$$\mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[\ell^{(i)}\right] = \alpha w_0, \tag{9}$$

where $\ell^{(i)}$ is defined in (8).[5]

---

[4]Here, with abuse of notation, we have used the same symbol to denote the limit of the individual liability under $\mathbb{P}$ in (7) and under $\widetilde{\mathbb{P}}$ in (8). If not explicitly stated, it will be clear from the context which expression should be considered.

[5]Convergence of the expectation in (8) can be justified through mild conditions on the log-return $R$.

Finally, taking the limit in the ruin condition (6) as $N_0 \to +\infty$ gives

$$\mathbb{P}\left(e^R \leq \alpha G \Pi\right) \leq \varepsilon. \tag{10}$$

The next Lemma establishes which combinations of leverage ratios $\alpha > 0$ and guaranteed returns $G$ will result in fair contracts. Note that direct inspection of the fairness condition (9) shows that it is always satisfied for the - uninteresting - case of *trivial contracts*, that is when $\alpha = 0$ (and $G$ is unrestricted), so that no insurance contract is formed. In the case of *full participation*, that is $\delta = 1$, the result shows that non trivial fair contracts necessarily require either that there is no stated guaranteed amount or that the entire capital is provided by policyholders (a mutual company, or a self supporting portfolio). In the case of *partial participation*, characterised by $\delta < 1$, then non trivial fair contracts are represented by a strictly increasing curve, as higher guaranteed level can be obtained by higher capital infusion by policyholders (more "skin in the game"), if fairness has to be preserved.

**Lemma 2.2.** *Let $0 < \alpha \leq 1$ and $G \geq 0$.*

    *i. If $\delta = 1$, then $(\alpha, G)$ satisfies (9) if and only if $\alpha = 1$ (and $G$ is unrestricted) or $G = 0$ (and $\alpha$ is unrestricted).*

    *ii. If $\delta < 1$ and $(\alpha, G)$ satisfies (9) then $\alpha < 1$. In particular, for each $0 < \alpha < 1$, there exists a unique $G(\alpha) > 0$ such that $G = G(\alpha)$.*

    *The function $G(\cdot) : (0,1) \to (0, +\infty)$ satisfies the following properties:*

      *(a) $G(\cdot)$ is strictly increasing.*

      *(b) Limits:*

$$\lim_{\alpha \to 1} G(\alpha) = +\infty, \tag{11}$$

$$\underline{G} := \lim_{\alpha \to 0} G(\alpha) > 0. \tag{12}$$

      *(c) $G(\cdot)$ can be extended to $[0,1)$ as a continuously differentiable function.*

Finally, the next Lemma reformulates the solvency constraint (10) and make explicit the inverse relation existing between the share contributed by policyholders and the guaranteed rate. Clearly, if the solvency criterion has to be satisfied, a higher guaranteed rate is consistent with lower contribution quotas by policyholders as this will lower the guaranteed amount.

**Lemma 2.3.** *For each $0 < \varepsilon < 1$, there exists $x_\varepsilon > 0$, depending on the distributions of $R$ and $\Pi$ under $\mathbb{P}$, such that (10) is equivalent to*

$$\alpha G \leq x_\varepsilon.$$

# 3. Optimal contracts

In this section we investigate how policyholders and equity holders contribute to the formation of a life insurance company. We consider a representative policyholder in a large portfolio maximizing her preferences by choosing the fraction of her wealth to be invested in the participating vehicle, while the insurer is obliged to offer fairly priced contracts and comply with regulatory solvency and minimum guarantee requirements. With reference to the setup described in the previous section, we fix a generic policyholder $i$ and assume her preferences are described through a twice differentiable utility function $u : \mathbb{R} \to \mathbb{R}$ such that $u' > 0$ and $u'' < 0$.

The policyholder decision problem is

$$
\max_{\alpha, G} \mathbb{E}\left[ u\left( \mathrm{e}^{-rT}\ell^{(i)} - \alpha w_0 \right) \right]
$$

$$
\text{s.t.} \quad
\begin{cases}
0 \leq \alpha \leq 1, \qquad 0 \leq G \leq \overline{G} \\
\mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[ \ell^{(i)} \right] = \alpha w_0 & \text{(fairness)} \\
\mathbb{P}\left( \mathrm{e}^R \leq \alpha G \Pi \right) \leq \varepsilon & \text{(solvency)}
\end{cases}
\tag{13}
$$

Hence, the policyholder aims at maximizing the expected utility of her net present value. However, note that

$$
\mathrm{e}^{-rT}\ell^{(i)} - \alpha w_0 = \mathrm{e}^{-rT}\left( \ell^{(i)} + (1-\alpha)w_0 \mathrm{e}^{rT} \right) - w_0,
$$

so that our problem is equivalent to optimally allocating the policyholder's initial wealth between the life insurance contract and the riskless asset.

We have implicitly assumed that the bargaining process between the two classes of stakeholders involves the fraction of capital provided by policyholders and the minimum guaranteed return, while other variables such as the participation rate, investment style and contract horizon have been fixed at the outset.

Inspection of the constraints in Problem (13), see Lemmas 2.2 and 2.3, shows that Problem (13) always has a solution.

**Remark 3.1.** *While the optimization problem has been framed from the point of view of the policyholder, it can actually be shown to span all non trivial contracts that are efficient, in a Pareto sense, for both stakeholders.*

*To see this, let $\mathcal{O}(\varepsilon)$ be the set of solutions of Problem (13) for any given ruin probability $0 < \varepsilon < 1$. We say that a couple $(\alpha, G)$ is a fair contract if $0 \leq \alpha \leq 1$, $0 \leq G \leq \overline{G}$ and (9) holds. Denoting*

$$
\mathcal{U}(\alpha, G) = \mathbb{E}\left[ u\left( \mathrm{e}^{-rT}\ell^{(i)} - \alpha w_0 \right) \right], \qquad \mathcal{R}(\alpha, G) = \mathbb{P}\left( \mathrm{e}^R \leq \alpha G \Pi \right),
$$

*a fair contract $(\alpha, G)$ is Pareto efficient if there exist no fair contracts $(\alpha', G')$ such that*

$$
\mathcal{U}(\alpha, G) \leq \mathcal{U}(\alpha', G'), \qquad \mathcal{R}(\alpha, G) \geq \mathcal{R}(\alpha', G'),
$$

*with at least one strict inequality.*

*It is then easily seen that*

$$
\{(\alpha, G) \text{ Pareto efficient contracts with } \alpha G > 0\} \subset \bigcup_{0 < \varepsilon < 1} \mathcal{O}(\varepsilon).
$$

Problem (13) might as well have as solution a couple $(\alpha, G)$ with $\alpha = 0$ (and $G$ is therefore irrelevant), because of excessively conservative pricing assumptions, inadequate bonus distribution policy or unattractive market returns. In these cases, policyholders are better off by allocating their entire wealth in the riskless asset. An example in this sense is given by the next proposition.

**Proposition 3.2.** *If there is no financial risk premium, that is $R$ has the same probability distribution under $\mathbb{P}$ as under $\widetilde{\mathbb{P}}$, then every solution $(\alpha^*, G^*)$ of Problem (13) satisfies $\alpha^* = 0$.*

This result has a classical flavour: if there is no compensation for financial risk and premiums are loaded for biometric risk, then any risk averse policyholder will prefer the risk-free investment to the (defaultable) participating insurance contract. Hence, from now on, we will assume that the stochastic dominance relation in Assumption (3) holds strictly, so that

$$\mathbb{E}[e^R] > \widetilde{\mathbb{E}}[e^R] = e^{rT}. \tag{14}$$

In order to investigate the shape of optimal contracts, we distinguish the case of full participation, $\delta = 1$, from that of partial participation, $\delta < 1$.

When a full participation policy is adopted by the insurer, the only non trivial fair contracts are those for which the entire capital is provided by policyholders, that is self supported portfolios, or those for which no return is guaranteed. The next result rules out the optimality of trivial contracts and provides a condition establishing whether the optimal capital share $\alpha^*$ is such that $\alpha^* < 1$ and hence the optimal guaranteed return is $G^* = 0$, or $\alpha^* = 1$.

**Theorem 3.3.** *When $\delta = 1$, then any solution $(\alpha^*, G^*)$ of Problem (13) is such that $\alpha^* > 0$. More precisely, $0 < \alpha^* < 1$ (and $G^* = 0$) if and only if*

$$\mathbb{E}\left[u'\left(w_0(J-1)\right)(J-1)\Pi - u'(-w_0)(1-\Pi)\right] < 0, \tag{15}$$

*where $J = \frac{e^{R-rT}}{\Pi}$. When (15) does not hold, then $\alpha^* = 1$ (and $G^*$ is any guaranteed return such that $G^* \leq \overline{G}$ and the solvency condition (10) holds).*

Focusing now on the case of partial participation, it turns out that no insurance may be optimal, e.g. when the participation rate is too low or the market risk premium is not high enough, or a combination of these conditions. Note that according to the fairness condition, the full contribution case $\alpha = 1$ is not viable.

**Theorem 3.4.** *The following statements hold:*

*(i) When $0 < \delta < 1$, there exists $\overline{\delta} = \overline{\delta}(\overline{G}) < 1$ with*

$$\overline{\delta} \begin{cases} > 0 & \text{if } \overline{G} < e^{rT}/\widetilde{\mathbb{E}}[\widetilde{\Pi}] \\ = 0 & \text{if } \overline{G} \geq e^{rT}/\widetilde{\mathbb{E}}[\widetilde{\Pi}] \end{cases},$$

*such that, for any solution $(\alpha^*, G^*)$ of (13),*

(a) *if* $0 < \delta \leq \overline{\delta}$ *then* $\alpha^* = 0$.

(b) *if* $\delta > \overline{\delta}$ *and*

$$\underline{G}\mathbb{E}\left[\Pi\right] + \delta\mathbb{E}\left[(e^R - \underline{G}\Pi)^+\right] > e^{rT}, \tag{16}$$

*where* $\underline{G} = \underline{G}(\delta)$ *is the quantity defined in Lemma 2.2, then* $\alpha^* > 0$ *(and* $G^* > 0$).

(ii) *When* $\delta = 0$ *and* $\overline{G} \leq \frac{e^{rT}}{\mathbb{E}[\Pi]}$ *then, for any solution* $(\alpha^*, G^*)$ *of* (13), $\alpha^* = 0$.

The constants $\overline{\delta}$ and $\underline{G}$ depend also on the risk free rate, the assets volatility, the contract maturity and the risk neutral distribution of the systematic risk factor $\widetilde{\Pi}$. From Theorem 3.4 it can be seen that, when no profits are redistributed to the policyholders, that is $\delta = 0$, then no insurance is the solution of Problem (13). More generally, as the function $\underline{G}(\cdot)$ is strictly decreasing, see Lemma 4.1 in Appendix 4, trivial contracts are optimal when the participation rate is not enough appealing for policyholders due to a too low maximum guaranteed return $\overline{G}$.

The next result establishes some conditions under which optimal policyholders contribution is positive.

**Corollary 3.5.** *Assume that* $\overline{\delta} < \delta < 1$, *where* $\overline{\delta}$ *has been introduced in Proposition 3.4, and let* $(\alpha^*, G^*)$ *be an optimal solution of Problem 13. If either*

i. $\delta$ *is sufficiently close to 1, or*

ii. $\Pi$ *under* $\mathbb{P}$ *and* $\widetilde{\Pi}$ *under* $\widetilde{\mathbb{P}}$ *have the same distribution,*

*then* $\alpha^* > 0$.

The result shows that condition (16) holds automatically when the participation coefficient is close enough to 100% or when there is no loading for (systematic) biometric risk.

# 4. Summary and Conclusions

In this paper we have studied the interaction between policyholders and equity holders in the formation of a life insurance company issuing participating contracts. Through the payment of a single premium each policyholder contributes to finance a fraction of the assets of the company and, in return, is entitled to receive a benefit at maturity, if alive. The structure of the contracts is stylized and features a guaranteed rate of return and a terminal bonus, as in the model by Briys and de Varenne (1994, 1997). We have considered an infinitely large population of perfectly homogeneous policyholders, where all the diversifiable longevity risk has been eliminated and only the systematic risk matters, along the lines of Bacinello et al. (2018). Policyholders aim at maximizing their preferences by choosing their rate of contribution (leverage ratio) and guaranteed return, while being subject to regulatory constraints of fair valuation, solvency and capped guarantee. We have provided conditions under which non trivial contracts exist and analyzed their properties. Moreover, we have shown that the solution of the policyholder's

optimization problem spans all non trivial contracts that are efficient, in a Pareto sense, for both policyholders and equity holders.

All our findings are analytical. Further research includes a numerical implementation of our theoretical model in order to explore the sensitivity of the results, in particular of the optimal solution of the problem, with respect to the various contract and market parameters.

# APPENDIX A
# Properties of Put and Call option prices

This Appendix establishes some (well or less well known) properties of option prices that are valid under the framework introduced in Section 2. In particular, the only assumption on the distribution of the log return $R$ is that its support is the entire real line. These properties will be used extensively in the proofs of results appearing in Section 4.

The initial prices of European call and put options with maturity $T$ and strike $K(\geq 0)$, written on the non-dividend paying risky asset with $R$ as log-return over the period $[0, T]$ and initial value $S(\geq 0)$, are given by

$$\text{Call}(S, K) = \widetilde{\mathbb{E}} \left[ e^{-rT}(Se^R - K)^+ \right],$$
$$\text{Put}(S, K) = \widetilde{\mathbb{E}} \left[ e^{-rT}(K - Se^R)^+ \right].$$

These option prices satisfy the *put-call parity* relation:

$$\text{Put}(S, K) = \text{Call}(S, K) - S + Ke^{-rT}.$$

Further, the option prices satisfy and the following inequalities:

$$
\begin{aligned}
&\max\{0, S - Ke^{-rT}\} \leq \text{Call}(S, K) \leq S, \\
&\max\{0, Ke^{-rT} - S\} \leq \text{Put}(S, K) \leq Ke^{-rT}, \\
&|\text{Call}(S, K_1) - \text{Call}(S, K_2)| \leq |K_1 - K_2| \, e^{-rT}, \\
&|\text{Put}(S, K_1) - \text{Put}(S, K_2)| \leq |K_1 - K_2| \, e^{-rT}.
\end{aligned}
\tag{A1}
$$

Moreover, the call option prices are increasing and convex with respect to the initial asset price $S$ and decreasing and convex with respect to the strike $K$, while the put prices are decreasing and convex with respect to $S$ and increasing and convex with respect to $K$. Consequently, the call and the put options are continuous with respect to both the initial asset price $S$ and the strike $K$.

In particular, taking into account that $R$ has as support the real line, if $S > 0$, $K > 0$ and $K_1 \neq K_2$, then all the inequalities in (A1), as well as the properties of monotonicity and convexity, are strict.

Finally, it is worth mentioning how option prices behave in the limiting cases for the asset value or the strike:

$$
\begin{aligned}
&\text{Call}(0, K) = 0, & &\lim_{S \to +\infty} \text{Call}(S, K) = +\infty, \\
&\text{Call}(S, 0) = S, & &\lim_{K \to +\infty} \text{Call}(S, K) = 0, \\
&\text{Put}(0, K) = Ke^{-rT}, & &\lim_{S \to +\infty} \text{Put}(S, K) = 0, \\
&\text{Put}(S, 0) = 0, & &\lim_{K \to +\infty} \text{Put}(S, K) = +\infty.
\end{aligned}
\tag{A2}
$$

# APPENDIX B
# A Study of the Fairness and Solvency Conditions

We first establish some equivalent versions of the fairness condition (9), which will be useful for the proofs of Lemmas 2.2 and 4.1.

For any $0 \leq \alpha \leq 1$ and $G \geq 0$, the initial value of the individual liability, appearing in the left hand side of (9), can be written as[6]

$$
\begin{aligned}
\mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[\ell^{(i)}\right] =& \widetilde{\mathbb{E}}\left[\mathrm{e}^{-rT}\alpha w_0 G\, 1_{E_i}\right] \\
&+ \widetilde{\mathbb{E}}\left[\mathrm{e}^{-rT}\delta\alpha w_0 \left(\frac{\mathrm{e}^R}{\widetilde{\Pi}} - G\right)^+ 1_{E_i}\right] \\
&- \widetilde{\mathbb{E}}\left[\mathrm{e}^{-rT} w_0 \left(\alpha G - \frac{\mathrm{e}^R}{\widetilde{\Pi}}\right)^+ 1_{E_i}\right] \\
=& \alpha w_0 G \mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] \\
&+ \delta\alpha w_0 \widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, G\widetilde{\Pi}\right)\right] \\
&- w_0 \widetilde{\mathbb{E}}\left[\mathrm{Put}\left(1, \alpha G\widetilde{\Pi}\right)\right].
\end{aligned}
$$

Using the latter expression, the fairness condition (9) can be equivalently written as

$$
\alpha = \alpha G\widetilde{\mathbb{E}}[\widetilde{\Pi}]\mathrm{e}^{-rT} + \delta\alpha\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, G\widetilde{\Pi}\right)\right] - \widetilde{\mathbb{E}}\left[\mathrm{Put}\left(1, \alpha G\widetilde{\Pi}\right)\right]. \tag{B1}
$$

When $\alpha > 0$, (B1) can be equivalently written as

$$
1 = G\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] + \delta\widetilde{\mathbb{E}}\left[\mathrm{Call}(1, G\widetilde{\Pi})\right] - \widetilde{\mathbb{E}}\left[\mathrm{Put}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right], \tag{B2}
$$

or, using the put-call parity, as

$$
1 - \delta = (1-\delta)G\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] + \delta\widetilde{\mathbb{E}}\left[\mathrm{Put}(1, G\widetilde{\Pi})\right] - \widetilde{\mathbb{E}}\left[\mathrm{Put}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right], \tag{B3}
$$

or as

$$
1 = \frac{1}{\alpha} + \delta\widetilde{\mathbb{E}}\left[\mathrm{Call}(1, G\widetilde{\Pi})\right] - \widetilde{\mathbb{E}}\left[\mathrm{Call}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right]. \tag{B4}
$$

**Proof of Lemma 2.2.** Note that (B1) always holds when $\alpha = 0$, for any value of $G$. In the remainder of the proof, assume that $\alpha > 0$ so that any of (B2), (B3) or (B4) can be used.

To show (1) ($\delta = 1$), note first that (B1) always holds when $G = 0$. Assuming then $G > 0$, Equation (B3) becomes

$$
\widetilde{\mathbb{E}}\left[\mathrm{Put}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right] = \widetilde{\mathbb{E}}\left[\mathrm{Put}(1, G\widetilde{\Pi})\right].
$$

---

[6]For the derivation of this expression, refer to a similar computation in the Appendix of Bacinello et al. (2018).

By strict monotonicity of the put option price, we conclude that the unique solution is $\alpha = 1$.

To show the first part of (2) ($0 \leq \delta < 1$), suppose that $\alpha = 1$. From (B4) one obtains

$$(1 - \delta)\widetilde{\mathbb{E}}\left[\mathrm{Call}(1, G\widetilde{\Pi})\right] = 0,$$

which is never verified because of positivity of the call.

Fix now $0 < \alpha < 1$. Defining the (continuous) function

$$\mathcal{C}(G) = 1 - \frac{1}{\alpha} + \widetilde{\mathbb{E}}\left[\mathrm{Call}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right] - \delta\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, G\widetilde{\Pi}\right)\right],$$

for $G \geq 0$, the fairness equation (B4) can be restated as $\mathcal{C}(G) = 0$. Note that

$$\mathcal{C}(0) = 1 - \delta > 0, \lim_{G \to +\infty} \mathcal{C}(G) = 1 - \frac{1}{\alpha} < 0.$$

If it is shown that $\mathcal{C}$ is strictly decreasing, then the claim follows. To see this, rewrite $\mathcal{C}$ as

$$\mathcal{C}(G) = 1 - \frac{1}{\alpha} + \overbrace{(1 - \delta)}^{>0}\overbrace{\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, G\widetilde{\Pi}\right)\right]}^{=c_1(G)}$$
$$+ \underbrace{\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right) - \mathrm{Call}\left(1, G\widetilde{\Pi}\right)\right]}_{=c_2(G)}.$$

Now, $c_1$ is strictly decreasing because of strict monotonicity of call prices with respect to the strike, while $c_2$ is strictly decreasing as it can be written as

$$c_2(G) = \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[H(G, \widetilde{\Pi})\right], \text{ with } H(G, p) = \widetilde{\mathbb{E}}\left[h(G, R, p)\right],$$

where

$$h(G, x, p) = \left(\frac{1}{\alpha}\mathrm{e}^x - Gp\right)^+ - \left(\mathrm{e}^x - Gp\right)^+.$$

Inspection of the function $h$ shows that it is decreasing in $G$ for any $x \in \mathbb{R}$ and $0 < p < 1$, implying that $H$ is strictly decreasing in $G$ for any $0 < p < 1$ and in turn that $c_2$ is strictly decreasing. This concludes the proof of the first part of (2).

Turning the attention on (2)(a), for $0 < \alpha < 1$ and $G \geq 0$, define the function

$$\mathcal{D}(\alpha, G) = 1 - G\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] - \delta\widetilde{\mathbb{E}}\left[\mathrm{Call}(1, G\widetilde{\Pi})\right] + \widetilde{\mathbb{E}}\left[\mathrm{Put}\left(\frac{1}{\alpha}, G\widetilde{\Pi}\right)\right],$$

so that the function $G(\cdot)$ is implicitly defined by $\mathcal{D}(\alpha, G(\alpha)) = 0$, see (B2). The function $\mathcal{D}$ is strictly decreasing in $G$ for any $0 < \alpha < 1$, indeed note that $\mathcal{C}(G) = \mathcal{D}(\alpha, G)$ for any $0 < \alpha < 1$. It is also clear that $\mathcal{D}$ is strictly increasing in $\alpha$ for any $G > 0$. Now, if $0 < \alpha' < \alpha'' < 1$ then $0 = \mathcal{D}(\alpha', G(\alpha')) < \mathcal{D}(\alpha'', G(\alpha'))$ and hence $G(\alpha'') > G(\alpha')$, showing the strict increasingness of $G(\cdot)$.

To establish (2)(b), take the limit in (B4) as $\alpha \to 1$. If $\lim_{\alpha \to 1} G(\alpha) = \widetilde{G} < +\infty$, then the equation

$$(1 - \delta)\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \widetilde{G}\widetilde{\Pi}\right)\right] = 0$$

should hold, a contradiction, due to the strict positivity of call prices.

Similarly, take the limit in (B2) as $\alpha \to 0$ to obtain the equation

$$1 - \underline{G}\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] = \delta\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \underline{G}\widetilde{\Pi}\right)\right], \tag{B5}$$

from which $\underline{G} > 0$.

Finally, part (2)(c) follows from a lengthy but straightforward application of the implicit function theorem. $\qquad\square$

The next result explores the behavior of the quantity $\underline{G} = G(0)$, that has been introduced in Lemma 2.2, as a function of the participation coefficient $\delta$.

**Lemma 4.1.** *Let $\delta < 1$. Writing $\underline{G} = \underline{G}(\delta)$ for the quantity defined in (12) as a function of $0 \le \delta < 1$, then*

    *i. $\underline{G}(\cdot)$ is strictly decreasing.*

    *ii. $\lim_{\delta \to 1} \underline{G}(\delta) = 0$.*

    *iii. $\underline{G}(\delta) \le \mathrm{e}^{rT}/\widetilde{\mathbb{E}}[\widetilde{\Pi}]$ for all $0 \le \delta < 1$ and equality holds if and only if $\delta = 0$.*

**Proof of Lemma 4.1.** Write $\underline{G} = \underline{G}(\delta)$ for $0 \le \delta < 1$. If $0 \le \delta' < \delta'' < 1$ and $\underline{G}(\delta'') \ge \underline{G}(\delta')$ then, using (B5) and the Lipschitz property of call option prices with respect to the strike, a contradiction is obtained:

$$\begin{aligned}
0 =& (\underline{G}(\delta'') - \underline{G}(\delta'))\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] + (\delta'' - \delta')\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \underline{G}(\delta')\widetilde{\Pi}\right)\right] \\
&+ \delta''\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \underline{G}(\delta'')\widetilde{\Pi}\right) - \mathrm{Call}\left(1, \underline{G}(\delta')\widetilde{\Pi}\right)\right] \\
\ge& (1 - \delta'')(\underline{G}(\delta'') - \underline{G}(\delta'))\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] + (\delta'' - \delta')\widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \underline{G}(\delta')\widetilde{\Pi}\right)\right] \\
>& 0.
\end{aligned}$$

Hence $\underline{G}(\cdot)$ is strictly decreasing, and (1) is proved.

Part (2) follows by taking the limit in (B5). Letting $\widetilde{G} = \lim_{\delta \to 1} \underline{G}(\delta)$, (B5) gives

$$1 - \widetilde{G}\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}] = \widetilde{\mathbb{E}}\left[\mathrm{Call}\left(1, \widetilde{G}\widetilde{\Pi}\right)\right].$$

By the lower bound for call options (A1), $\widetilde{\mathbb{E}}[\mathrm{Call}(1, \widetilde{G}\widetilde{\Pi})] \ge 1 - \widetilde{G}\mathrm{e}^{-rT}\widetilde{\mathbb{E}}[\widetilde{\Pi}]$ and the equality holds if and only if $\widetilde{G} = 0$, we deduce that $\widetilde{G} = 0$.

Part (3) immediately follows from (B5), upon remarking that the right-hand side is non-negative and null if and only if $\delta = 0$. $\qquad\square$

The next proof refers to the solvency condition (10).

**Proof of Lemma 2.3.** Constraint (10) can be equivalently written as

$$\mathcal{T}(\alpha G) \le \varepsilon,$$

where, for $x > 0$,

$$\mathcal{T}(x) = \mathbb{E}\Big[F_R\big(\log(x\Pi)\big)\Big]$$

and $F_R(\cdot)$ is the (strictly increasing) cumulative distribution function of the assets log-return $R$ under $P$.

The function $\mathcal{T}$ is continuous, strictly increasing and satisfies $\lim_{x \to 0^+} \mathcal{T}(x) = 0$, $\lim_{x \to +\infty} \mathcal{T}(x) = 1$. The claim follows by letting $x_\varepsilon > 0$ be the unique solution of $\mathcal{T}(x_\varepsilon) = \varepsilon$. $\qquad\qquad\square$

# APPENDIX C
# Proofs of results in Section 3

We write, for $(\alpha, G)$ with $0 \le \alpha \le 1$ and $G \ge 0$,

$$\mathcal{U}(\alpha, G) = \mathbb{E}\left[u\left(e^{-rT}\ell^{(i)} - \alpha w_0\right)\right].$$

It follows from the tower property and the independence between financial and biometric risk (under $\mathbb{P}$) that

$$\mathcal{U}(\alpha, G) = \mathbb{E}\left[\Pi u(h_1) + (1 - \Pi)u(h_2)\right], \tag{C1}$$

where

$$h_1 = w_0 \left(e^{-rT}\left(\alpha G + \delta\alpha\left(\tfrac{e^R}{\Pi} - G\right)^+ - \left(\alpha G - \tfrac{e^R}{\Pi}\right)^+\right) - \alpha\right),$$
$$h_2 = -\alpha w_0.$$

**Proof of Proposition 3.2.** For $(\alpha, G)$ feasible for Problem (13) with $\alpha > 0$, we have

$$
\begin{aligned}
\mathcal{U}(\alpha, G) &= \mathbb{E}\left[\Pi u(h_1) + (1 - \Pi)u(h_2)\right] \\
&< \mathbb{E}\left[u(\Pi h_1 + (1 - \Pi)h_2]\right] && u \text{ strictly concave, } h_1 \ne h_2 \\
& && \text{and } 0 < \Pi < 1 \text{ a.s.} \\
&< u\left(\mathbb{E}\left[\Pi h_1 + (1 - \Pi)h_2\right]\right) && \text{Jensen inequality,} \\
& && \Pi h_1 + (1 - \Pi)h_2 \text{ non degenerate} \\
&= u(\mathbb{E}\left[g(\Pi, R)\right]),
\end{aligned}
$$

where

$$g(p, x) = w_0 e^{-rT}\left(\alpha G p + \delta\alpha\left(e^x - Gp\right)^+ - \left(\alpha Gp - e^x\right)^+\right) - \alpha w_0.$$

We next prove that

$$\mathbb{E}\left[g(\Pi, R)\right] \le \widetilde{\mathbb{E}}\left[g(\widetilde{\Pi}, R)\right] = 0, \tag{C2}$$

so that the conclusion follows as

$$\mathcal{U}(\alpha, G) < u(\mathbb{E}\big[g(\Pi, R)\big]) \leq u(0) = \mathcal{U}(0, \widetilde{G})$$

for any $0 \leq \widetilde{G} \leq \overline{G}$, so that no insurance is optimal.

We first prove the inequality in (C2). Note that, using the tower property and the independence between financial and biometric risk (under $\mathbb{P}$),

$$\mathbb{E}\left[g(\Pi, R)\right] = \mathbb{E}\left[f(\Pi)\right],$$

where $f(p) = \mathbb{E}\left[g(p, R)\right]$, and similarly

$$\widetilde{\mathbb{E}}\left[g(\widetilde{\Pi}, R)\right] = \widetilde{\mathbb{E}}\left[\widetilde{f}(\widetilde{\Pi})\right],$$

where $\widetilde{f}(p) = \widetilde{\mathbb{E}}\left[g(p, R)\right]$. Now, $f = \widetilde{f}$ a.e. as the laws of $R$ under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ coincide. Inspection of the function $g(p, x)$ shows that it is increasing in $p$ for any $x \in \mathbb{R}$. This implies that $f$ is increasing and hence, using Assumption (4),

$$\mathbb{E}\left[g(\Pi, R)\right] = \mathbb{E}\left[f(\Pi)\right] \leq \widetilde{\mathbb{E}}\left[f(\widetilde{\Pi})\right] = \widetilde{\mathbb{E}}\left[g(\widetilde{\Pi}, R)\right].$$

Finally, to show the equality in (C2), notice that the fairness equation (9) can be restated, using an argument similar to that leading to (C1), as

$$\mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[\ell^{(i)}\right] = \alpha w_0$$
$$\Leftrightarrow \widetilde{\mathbb{E}}\left[\mathrm{e}^{-rT}\ell^{(i)} - \alpha w_0\right] = 0.$$
$$\Leftrightarrow \widetilde{\mathbb{E}}\left[g(\widetilde{\Pi}, R)\right] = 0.$$

$\square$

**Proof of Theorem 3.3.**

According to Lemma 2.2, when $\delta = 1$, the only fair contracts $(\alpha, G)$ are those for which either $\alpha = 0$ or $\alpha = 1$ or $G = 0$. Also, note that, when $\alpha = 0$ or $\alpha = 1$, the expected utility does not depend on $G$. Therefore, we study the function $g(\alpha) = \mathcal{U}(\alpha, 0)$ for $0 \leq \alpha \leq 1$. A straightforward calculation gives

$$g'(\alpha) = w_0 \mathbb{E}\left[u'\left(\alpha w_0(J-1)\right)(J-1)\Pi - u'(-\alpha w_0)(1-\Pi)\right].$$

and, in particular, (14) implies

$$g'(0) = w_0 u'(0)\mathbb{E}\left[\mathrm{e}^{R-rT} - 1\right] > 0,$$

so that the optimal $\alpha$ is positive. Furthermore, $g$ is strictly concave since

$$g''(\alpha) = w_0^2 \mathbb{E}\left[u''\left(\alpha w_0(J-1)\right)(J-1)^2\Pi + u''(-\alpha w_0)(1-\Pi)\right] < 0.$$

Finally, $\alpha = 1$ is optimal if and only if

$$g'(1) = w_0 \mathbb{E}\left[u'\left(w_0(J-1)\right)(J-1)\Pi - u'(-w_0)(1-\Pi)\right] \geq 0,$$

so the claim follows. □

**Proof of Theorem 3.4.**

Using the results in Lemma 4.1 (1), when $\overline{G} < \mathrm{e}^{rT}/\widetilde{\mathbb{E}}[\widetilde{\Pi}]$ we define $\overline{\delta} > 0$ as the unique solution of $\underline{G}(\overline{\delta}) = \overline{G}$ and set $\overline{\delta} = 0$ when instead $\overline{G} \geq \mathrm{e}^{rT}/\widetilde{\mathbb{E}}[\widetilde{\Pi}]$.

When $0 \leq \delta \leq \overline{\delta}$, Lemma 4.1 (1) implies that $\underline{G}(\delta) \geq \overline{G}$. Therefore, by Lemmas 2.2 (2) and 4.1 (1,3), we conclude that any fair contract $(\alpha, G)$ must satisfy $\alpha = 0$.

Assume then $\overline{\delta} < \delta$. By Lemmas 2.2 (2), 4.1 and 2.3 there is an interval $[0, \overline{\alpha}]$ such that the fair contract $(\alpha, G(\alpha))$ satisfies the solvency constraint for any $0 \leq \alpha \leq \overline{\alpha}$.

Consider now the function

$$H(\alpha) = \mathcal{U}(\alpha, G(\alpha)) = \mathbb{E}\left[u\left(h_1(\alpha)\right)\Pi + u\left(h_2(\alpha)\right)(1 - \Pi)\right],$$

where $h_1(\alpha), h_2(\alpha)$ are defined as in equation (C1) with $G = G(\alpha)$, and the dependence on $\alpha$ has been made explicit.

Differentiating this function gives, after some algebra,

$$
\begin{aligned}
H'(\alpha) =& \mathbb{E}\left[u'\left(h_1(\alpha)\right)h_1'(\alpha)\Pi + u'\left(h_2(\alpha)\right)h_2'(\alpha)(1 - \Pi)\right] \\
=& w_0 \mathbb{E}\Bigg[u'\left(h_1(\alpha)\right)\left(\mathrm{e}^{-rT}\Big\{\partial_\alpha(\alpha G(\alpha))1_{\{R > \log(\alpha G(\alpha)\Pi)\}}\right. \\
& + \delta\left(\left(\tfrac{\mathrm{e}^R}{\Pi} - G(\alpha)\right)^+ - \alpha G'(\alpha)1_{\{R > \log(G(\alpha)\Pi)\}}\right)\Big\} - 1\Bigg)\Pi \\
& \left. - u'\left(h_2(\alpha)\right)(1 - \Pi)\right].
\end{aligned}
$$

Noting that $h_j(0) = 0$, for $j = 1, 2$, we have

$$H'(0) = w_0 u'(0)\left(\mathrm{e}^{-rT}\left(\underline{G}(\delta)\mathbb{E}\left[\Pi\right] + \delta\mathbb{E}\left[(\mathrm{e}^R - \underline{G}(\delta)\Pi)^+\right]\right) - 1\right),$$

and the claim follows. □

**Proof of Corollary 3.5.** We first prove (1). Note that

$$\lim_{\delta \to 1}\left(\underline{G}\mathbb{E}\left[\Pi\right] + \delta\mathbb{E}\left[(\mathrm{e}^R - \underline{G}\Pi)^+\right]\right) = \mathbb{E}\left[\max\{\mathrm{e}^R, \underline{G}(1)\Pi\}\right] \geq \mathbb{E}\left[\mathrm{e}^R\right] > \mathrm{e}^{rT},$$

where the last inequality follows from (14). Consequently, there exists $\delta'$ as in (1), such that, for all $\delta' < \delta < 1$, (16) holds. For all such $\delta$, applying Theorem 3.4, we obtain (1).

To show (2), consider $\overline{\delta} < \delta < 1$ and recall equation (B5):

$$1 = \mathrm{e}^{-rT}\left(\underline{G}\widetilde{\mathbb{E}}[\widetilde{\Pi}] + \delta\widetilde{\mathbb{E}}\left[(\mathrm{e}^R - \underline{G}\widetilde{\Pi})^+\right]\right).$$

By assumption, $\widetilde{\mathbb{E}}[\widetilde{\Pi}] = \mathbb{E}[\Pi]$. We next show that

$$\widetilde{\mathbb{E}}\left[(\mathrm{e}^R - \underline{G}\widetilde{\Pi})^+\right] < \mathbb{E}\left[(\mathrm{e}^R - \underline{G}\Pi)^+\right], \tag{C3}$$

so that (16) is obtained and the claim holds. The proof follows the argument exploited in the proof of Proposition 3.2. For the sake of completeness, we provide it here in full

detail. Letting $m(p, x) = (e^x - \underline{G}p)^+$, an increasing function of $x$ for any $0 \leq p \leq 1$, condition (C3) can be equivalently written as

$$\widetilde{\mathbb{E}}\big[m(\widetilde{\Pi}, R)\big] < \mathbb{E}\left[m(\Pi, R)\right].$$

Let us set $\tilde{n}(x) = \widetilde{\mathbb{E}}[m(\widetilde{\Pi}, x)]$, which is increasing for any $0 \leq p \leq 1$. By the assumption on $\widetilde{\Pi}$ under $\widetilde{\mathbb{P}}$ and $\Pi$ under $\mathbb{P}$, it is also $\tilde{n}(x) = \mathbb{E}[m(\Pi, x)]$. By the tower property and the independence between financial and biometric risks, we get

$$\widetilde{\mathbb{E}}\big[m(\widetilde{\Pi}, R)\big] = \widetilde{\mathbb{E}}\left[\tilde{n}(R)\right].$$

Similarly,

$$\mathbb{E}\left[m(\Pi, R)\right] = \mathbb{E}\left[\tilde{n}(R)\right].$$

Using Assumption (3), we conclude

$$\widetilde{\mathbb{E}}\big[m(\widetilde{\Pi}, R)\big] = \widetilde{\mathbb{E}}\big[\tilde{n}(R)\big] < \mathbb{E}\big[\tilde{n}(R)\big] = \mathbb{E}\left[m(\Pi, R)\right].$$

$\square$

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